Homework 6 Solutions

8.1.2) This system has fixed points at

$$x^* = \pm\sqrt{\mu} \text{ and } y^* = 0.$$

The Jacobian matrix is:

$$A = \begin{bmatrix} -2x & 0 \\ 0 & -1 \end{bmatrix}.$$

This has eigenvalues $\lambda = -2x, -1$. So, the stable fixed point occurs at $x^* = \sqrt{\mu}$.

Since $\lambda_1 = -2\sqrt{\mu}$, we see that $\lambda_1 \to 0$ as $\mu \to 0$.

8.1.3) This system has fixed points at

$$x^* = 0, \mu \text{ and } y^* = 0.$$

The Jacobian matrix is:

$$A = \begin{bmatrix} \mu - 2x & 0 \\ 0 & -1 \end{bmatrix}.$$

This has eigenvalues $\lambda = \mu - 2x, -1$.
So, the stable fixed occurs at $x^* = \mu$ if $\mu > 0$ and at $x^* = 0$ if $\mu < 0$.

When $\mu > 0$: $\lambda_1 = -\mu \to 0$ as $\mu \to 0$.

When $\mu < 0$: $\lambda_1 = \mu \to 0$ as $\mu \to 0$.

8.1.4) This system has fixed points at $x^* = 0, \pm\sqrt{-\mu}$ and $y^* = 0$.

The Jacobian matrix is:

$$A = \begin{bmatrix} \mu + 3x^2 & 0 \\ 0 & -1 \end{bmatrix}.$$

This has eigenvalues $\lambda = \mu + 3x^2, -1$.

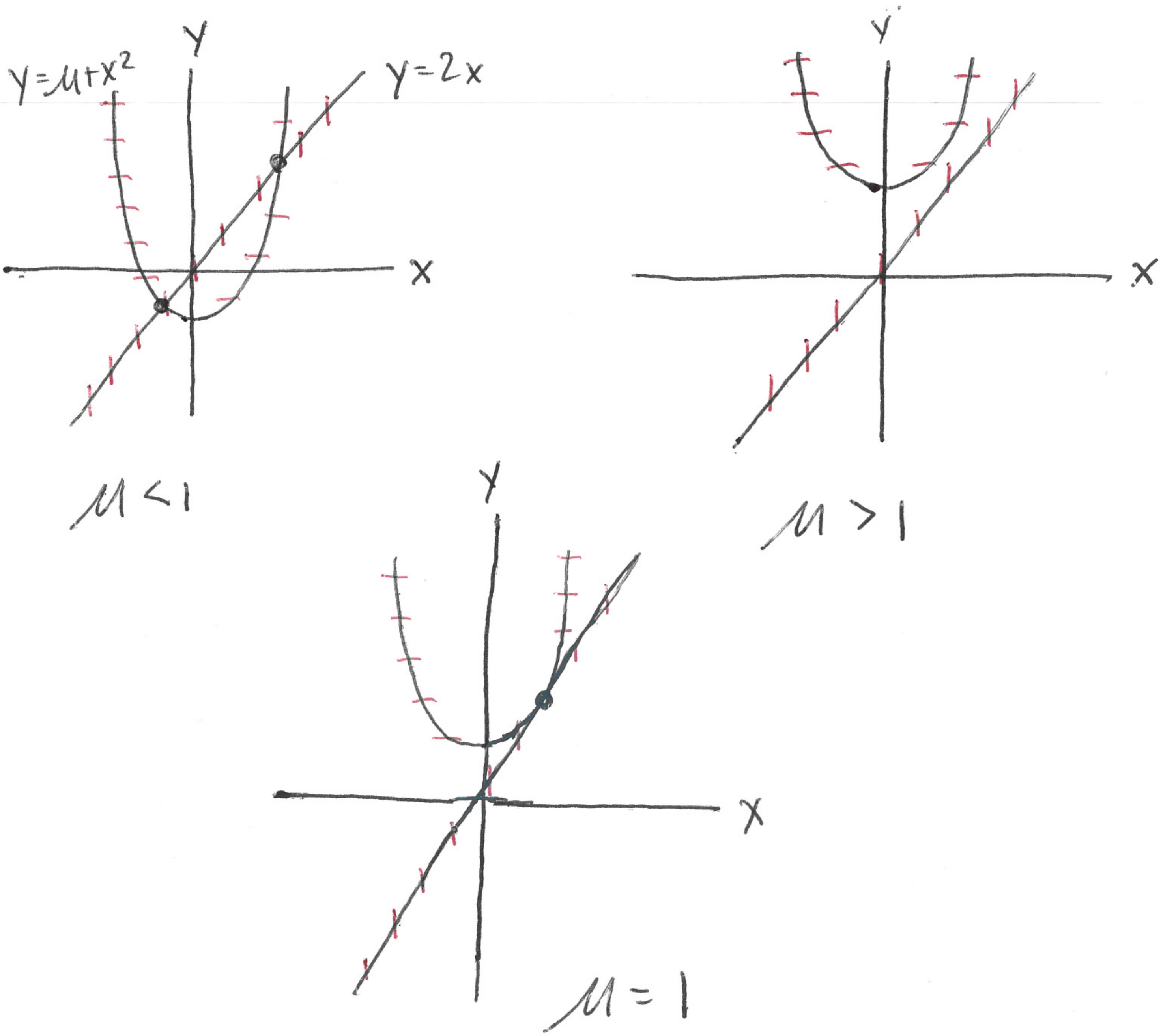
The only stable fixed point occurs at $x^* = 0$ if $\mu < 0$.

Since $\lambda_1 = \mu$, we see that $\lambda_1 \to 0$ as $\mu \to 0$.

8.1.6) a) The nullclines are given by the curves $y - 2x = 0$, $\mu + x^2 - y = 0$.

Alternatively: $y = 2x$, $y = \mu + x^2$.

These two curves are tangent when $\mu = 1$.



b) There is one bifurcation at $\mu = 1$, which is clearly a saddle-node bifurcation.

8.2.2) We find the Jacobian matrix:

$$A = \begin{bmatrix} \mu + y^2 & -1 + 2xy \\ 1 - 2x & \mu \end{bmatrix}.$$

Setting $x = y = \mu = 0$, we find:

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

This has eigenvalues $\lambda = \pm i$.

8.2.3) You should find that the origin is a spiral source for $\mu > 0$ and a spiral sink encircled by an unstable limit cycle for $\mu < 0$. This is a subcritical Hopf bifurcation.

8.2.4) We have:

$$\dot{r} = \frac{1}{r}(x\dot{x} + y\dot{y})$$

$$= \frac{1}{r}\Big(x(\cancel{-y} + \mu x + xy^2) + y(\cancel{x} + \mu y - x^2)\Big)$$

$$= \frac{1}{r}\left(\mu(x^2+y^2) + x^2y^2 - x^2y\right)$$

$$= \frac{1}{r}\left(\mu r^2 + r^4\cos^2\theta\sin^2\theta - r^3\cos^2\theta\sin\theta\right)$$

$$= \mu r + r^3\cos^2\theta\sin^2\theta - r^2\cos^2\theta\sin\theta.$$

$$\dot{\theta} = \frac{1}{r^2}(x\dot{y} - y\dot{x})$$

$$= \frac{1}{r^2}\Big(x(x + \cancel{\mu y} - x^2) - y(-y + \cancel{\mu x} + xy^2)\Big)$$

$$= \frac{1}{r^2}\left(x^2 + y^2 - x^3 - xy^3\right)$$

$$= \frac{1}{r^2}\left(r^2 - r^3\cos^3\theta - r^4\cos\theta\sin^3\theta\right)$$

$$= 1 - r\cos^3\theta - r^2\cos\theta\sin^3\theta.$$

b) If $r \ll 1$, we can disregard terms in $\dot{\theta}$ proportional to r. So $\dot{\theta} \approx 1$.

We also average over one revolution in $\dot{r}$ to find:

$$\dot{r} \approx \frac{1}{2\pi}\int_0^{2\pi}\left(\mu r + r^3\cos^2\theta\sin^2\theta - r^2\cos^2\theta\sin\theta\right)d\theta$$

averages to zero

$$= \mu r + \frac{r^3}{2\pi}\int_0^{2\pi}\cos^2\theta\sin^2\theta\, d\theta$$

$$= \mu r + \frac{r^3}{2\pi}\left(\frac{\pi}{4}\right) = \mu r + \frac{1}{8}r^3.$$