Homework 5 Solutions

7.1.5) We have $x = r\cos\theta$ and $y = r\sin\theta$,

so:

$$\dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$$
$$= \frac{x\dot{r}}{r} - y\dot{\theta}$$
$$= x(1-r^2) - y$$
$$= x - y - x(x^2+y^2),$$

$$\dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$$
$$= \frac{y\dot{r}}{r} + x\dot{\theta}$$
$$= y(1-r^2) + x$$
$$= x + y - y(x^2+y^2).$$

7.1.8) a) Write the system as:

$$\dot{x} = y, \quad \dot{y} = -ay(x^2+y^2-1) - x,$$

The only fixed point is $(x^*, y^*) = (0,0)$.

At $(0,0)$:

$$A = \begin{bmatrix} 0 & 1 \\ -2axy-1 & -a(x^2+3y^2-1) \end{bmatrix} \Bigg|_{(0,0)}$$

$$= \begin{bmatrix} 0 & 1 \\ -1 & a \end{bmatrix} \Rightarrow \lambda_{1,2} = \frac{a \pm \sqrt{a^2-4}}{2}$$

$\Rightarrow$ Unstable spiral if $0 < a < 2$,
Unstable node if $a > 2$.

b) Change to polar coordinates:

$$r\dot{r} = x\dot{x} + y\dot{y}$$

$$= y\left(-ay(x^2+y^2-1) - x\right) + xy$$

$$= -ay^2(x^2+y^2-1)$$

$$= -ar^2\sin^2\theta\,(r^2-1)$$

$$\Rightarrow \dot{r} = -ar\sin^2\theta\,(r^2-1).$$

$$\dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^2}$$

$$= \frac{x(-ay(x^2+y^2-1)-x) - y^2}{r^2}$$

$$= \frac{-x^2-y^2-axy(x^2+y^2-1)}{r^2}$$

$$= \frac{-r^2 - ar^2 \sin\theta\cos\theta(r^2-1)}{r^2}$$

$$= -1 - a\sin\theta\cos\theta(r^2-1).$$

Thus, there is a circular limit cycle with amplitude 1, on which $\dot{\theta} = -1$, so the period is 2π.

c) If $f(r,\theta) = -ar\sin^2\theta(r^2-1)$, then:

$$\left.\frac{\partial f}{\partial r}\right|_{r=1} = -a\sin^2\theta(3r^2-1)\Big|_{r=1}$$

$$= -2a\sin^2\theta \leq 0.$$

Thus, the limit cycle is stable.

7.2.5) a) If this is a gradient system, then there is a scalar function $V(x,y)$ so that:

$$f(x,y) = -\frac{\partial V}{\partial x}, \quad g(x,y) = -\frac{\partial V}{\partial y}.$$

Thus:

$$\frac{\partial f}{\partial y} = -\frac{\partial}{\partial y}\frac{\partial V}{\partial x} = -\frac{\partial}{\partial x}\frac{\partial V}{\partial y} = \frac{\partial g}{\partial x}.$$

b) Yes, so long as f and g are smooth.

7.2.7) a) We have:

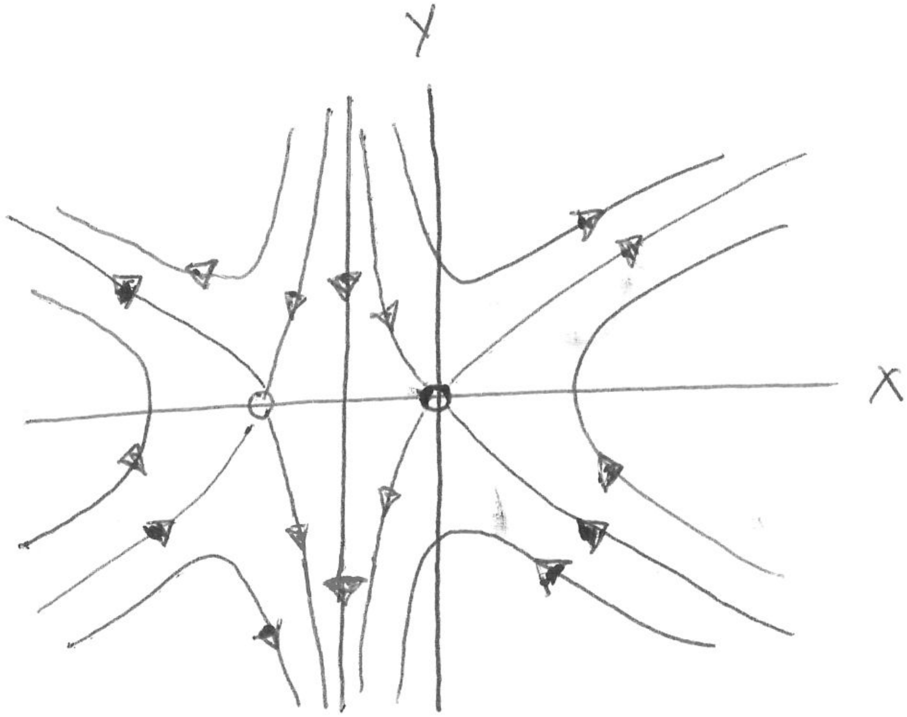
$$\frac{\partial f}{\partial y} = 1+2x, \quad \frac{\partial g}{\partial x} = 1+2x.$$

Thus, $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$.

b) It is straightforward to find

$$V(x,y) = \frac{1}{3}y^3 - xy - x^2 y.$$

c)



7.3.1) a) We have:

$$A = \begin{bmatrix} 1-3x^2-5y^2 & -1-10xy \\ 1-2xy & 1-3y^2-x^2 \end{bmatrix}\Bigg|_{(0,0)}$$

$$= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \Rightarrow \lambda_{1,2} = 1 \pm i$$

$\Rightarrow$ Unstable spiral.

b) $r\dot{r} = x\dot{x} + y\dot{y} = x\left(x-y-x(x^2+5y^2)\right)$
$+y\left(x+y-y(x^2+y^2)\right)$

$$= x^2+y^2-x^2(x^2+5y^2)-y^2(x^2+y^2)$$

$$= x^2+y^2-(x^2+y^2)^2-4x^2y^2$$

$$= r^2-r^4-4r^4\cos^2\theta\sin^2\theta$$

$$= r^2-r^4(1+4\cos^2\theta\sin^2\theta)$$

$$\Rightarrow \dot{r} = r-r^3(1+4\cos^2\theta\sin^2\theta).$$

$$r^2\dot{\theta} = x\dot{y}-y\dot{x}$$

$$= x(x+y-y(x^2+y^2))$$

$$-y(x-y-x(x^2+5y^2))$$

$$= x^2+y^2+4xy^3$$

$$= r^2+4r^4\cos\theta\sin^3\theta$$

$$\Rightarrow \dot{\theta} = 1+4r^2\cos\theta\sin^3\theta.$$

c) We have $1+4\cos^2\theta\sin^2\theta \le 5$,

So:

$$\dot{r} = r - r^3(1+4\cos^2\theta\sin^2\theta)$$

$$\ge r - 5r^3$$

If we require $\dot{r} \ge 0$, then:

$$r - 5r^3 \ge 0 \Rightarrow r \le 1/\sqrt{5}.$$

So, $r_1 = 1/\sqrt{5}$.

d) We have $1+4\cos^2\theta\sin^2\theta \ge 1$,

So:

$$\dot{r} = r - r^3(1+4\cos^2\theta\sin^2\theta)$$

$$\le r - r^3$$

If we require $\dot{r} \le 0$, then:

$$r - r^3 \le 0 \Rightarrow r \ge 1.$$

Thus $r_2 = 1$.

d) This should be clear, since all assumptions of the Poincaré-Bendixson theorem are satisfied.

7.6.1) We have:

$$x(t,\varepsilon) = (1-\varepsilon^2)^{-1/2} e^{-\varepsilon t} \sin\left((1-\varepsilon^2)^{1/2} t\right)$$

$$\begin{aligned}\frac{\partial x}{\partial \varepsilon}(t,\varepsilon) = \; & \varepsilon(1-\varepsilon^2)^{-3/2} e^{-\varepsilon t} \sin\left((1-\varepsilon^2)^{1/2} t\right) \\ & - t(1-\varepsilon^2)^{-1/2} e^{-\varepsilon t} \sin\left((1-\varepsilon^2)^{1/2} t\right) \\ & - t\varepsilon(1-\varepsilon^2)^{-1} e^{-\varepsilon t} \cos\left((1-\varepsilon^2)^{1/2} t\right)\end{aligned}$$

$$\Rightarrow x(t,0) = \sin(t),$$

$$\frac{\partial x}{\partial \varepsilon}(t,\varepsilon) = -t \sin t.$$

$$\Rightarrow x(t,\varepsilon) = \sin t - \varepsilon t \sin t + \mathcal{O}(\varepsilon^2).$$

7.6.2) a) $x(t,\varepsilon) = \cos\left(t\sqrt{1+\varepsilon}\right)$.

b) Plugging into the ODE:

$$\left(\ddot{x}_0 + \varepsilon \ddot{x}_1 + \varepsilon^2 \ddot{x}_2 + O(\varepsilon^3)\right)$$
$$+ \left(\dot{x}_0 + \varepsilon \dot{x}_1 + \varepsilon^2 \dot{x}_2 + O(\varepsilon^3)\right)$$
$$+ \varepsilon\left(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + O(\varepsilon^3)\right) = 0$$

At $O(1)$:

$$\ddot{x}_0 + \dot{x}_0 = 0$$

At $O(\varepsilon)$:

$$\ddot{x}_1 + \dot{x}_1 + x_0 = 0$$

At $O(\varepsilon^2)$:

$$\ddot{x}_2 + \dot{x}_2 + x_1 = 0$$

With boundary conditions:

$x_0(0) = 1$, $x_1(0) = 0$, $x_2(0) = 0$,

$\dot{x}_0(0) = 0$, $\dot{x}_1(0) = 0$, $\dot{x}_2(0) = 0$.

Solving the ODEs at $O(1)$, $O(\varepsilon)$, and $O(\varepsilon^2)$ yields:

$$x_0(t) = \cos t,$$

$$x_1(t) = -\tfrac{1}{2} t \sin t$$

$$x_2(t) = -\tfrac{1}{8} t^2 \cos t - \tfrac{1}{8} t \sin t$$

c) Yes, because the term ε introduces a slow timescale due to the frequency shift.

7.6.3) a) $x(t, \varepsilon) = \varepsilon + (1-\varepsilon)\cos t.$

b) Plugging into the ODE:

$$(\ddot{x}_0 + \varepsilon \ddot{x}_1 + \varepsilon^2 \ddot{x}_2 + O(\varepsilon^3))$$
$$+ (x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + O(\varepsilon^3)) = \varepsilon$$

At $O(1)$: $\ddot{x}_0 + x_0 = 0$

At $O(\varepsilon)$: $\ddot{x}_1 + x_1 = 1$

At $O(\varepsilon^2)$: $\ddot{x}_2 + x_2 = 0$

With boundary conditions:

$$x_0(0) = 1, \quad x_1(0) = 0, \quad x_2(0) = 0,$$

$$\dot{x}_0(0) = 0, \quad \dot{x}_1(0) = 0, \quad \dot{x}_2(0) = 0.$$

Solving these ODEs yields:

$$x_0(t) = \cos t,$$

$$x_1(t) = 1 - \cos t,$$

$$x_2(t) = 0.$$

c) No, there is only a single time scale, and in fact the solution is a power series in ε.