Homework 4 Solutions

6.2.1) No. Solutions approach a fixed point arbitrarily closely but do not reach it in finite time. Thus, the trajectories do not truly intersect.

6.2.2) a) It should be clear that $f(x,y) = y$ and $g(x,y) = -x + (1-x^2-y^2)y$ are continuous, and their partial derivatives are continuous, on D since they are polynomials.

b) We have:

$$y = \cos t = \dot{x},$$

$$\begin{aligned} -x + (1-x^2-y^2)y &= -\sin t + (1 - \underbrace{\sin^2 t - \cos^2 t}_{=1})\cos t \\ &= -\sin t \\ &= \dot{y}. \end{aligned}$$

c) The solution starts within the circle $x^2+y^2=1$, which is the trajectory of the solution from (b). It cannot cross this trajectory, therefore it must satisfy $x^2+y^2<1$.

6.3.1) This system has fixed points at $(x^*, y^*) = (2,2), (-2,-2)$.

At $(2,2)$:

$$A = \left.\begin{bmatrix} 1 & -1 \\ 2x & 0 \end{bmatrix}\right|_{(2,2)} = \begin{bmatrix} 1 & -1 \\ 4 & 0 \end{bmatrix}$$

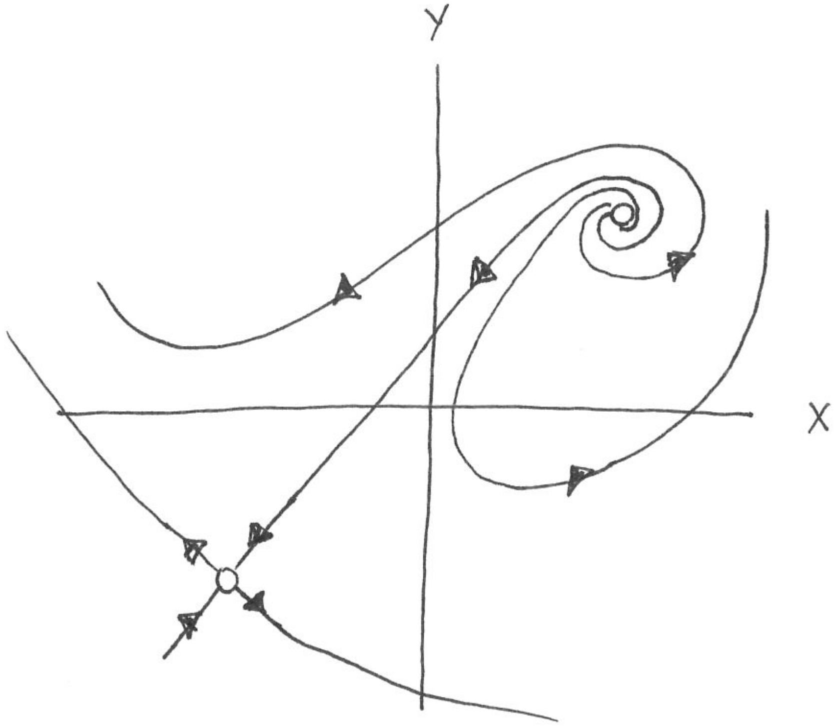
$$\Rightarrow \lambda_{1,2} = \frac{1 \pm i\sqrt{15}}{2} \Rightarrow \text{Unstable Spiral}$$

At $(-2,-2)$:

$$A = \begin{bmatrix} 1 & -1 \\ -4 & 0 \end{bmatrix} \Rightarrow \lambda_{1,2} = \frac{1 \pm \sqrt{17}}{2} \approx 2.6, -1.6$$

$$\Rightarrow \bar{V}_{1,2} = \begin{bmatrix} \frac{-1 \mp \sqrt{17}}{8} \\ 1 \end{bmatrix} \approx \begin{bmatrix} -0.6 \\ 1 \end{bmatrix}, \begin{bmatrix} 0.4 \\ 1 \end{bmatrix}$$

$$\Rightarrow \text{Saddle Point}$$



6.3.11) a) We have:

$$\dot{r} = -r \Rightarrow r = r_0 e^{-t},$$

$$\dot{\theta} = \frac{1}{\ln r} = \frac{1}{\ln(r_0 e^{-t})} = \frac{1}{\ln r_0 - t} \Rightarrow \theta = \theta_0 + \ln\left(\frac{\ln r_0}{\ln r_0 - t}\right).$$

b) So long as $r_0 < 1$, we have $r \to 0$ clearly, and:

$$\frac{\ln r_0}{\ln r_0 - t} \longrightarrow \quad \Rightarrow \theta = \theta_0 + \ln\left(\frac{\ln r_0}{\ln r_0 - t}\right) \to \infty$$

as $t \to \infty$.

c) If $x = r\cos\theta$ and $y = r\sin\theta$,

We have:

$$\dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta,$$

$$\dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta.$$

So:

$$\dot{x} = -r\cos\theta - \frac{r\sin\theta}{\ln r}$$

$$= -x - \frac{y}{\ln(x^2+y^2)},$$

$$\dot{y} = -r\sin\theta + \frac{r\cos\theta}{\ln r}$$

$$= -y + \frac{x}{\ln(x^2+y^2)}$$

d) We have:

$$A = \begin{bmatrix} -1 + \frac{2xy}{(x^2+y^2)\ln^2(x^2+y^2)} & -\frac{1}{\ln(x^2+y^2)} + \frac{2y^2}{(x^2+y^2)\ln^2(x^2+y^2)} \\ \frac{1}{\ln(x^2+y^2)} - \frac{2x^2}{(x^2+y^2)\ln^2(x^2+y^2)} & -1 - \frac{2xy}{(x^2+y^2)\ln^2(x^2+y^2)} \end{bmatrix}$$

Taking the limit $(x,y) \to (0,0)$, we find:

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

6.3.12) By the chain rule:

$$\dot{\theta} = \frac{-\dot{x}\, y}{x^2(1+y^2/x^2)} + \frac{\dot{y}}{x(1+y^2/x^2)}$$

$$= \frac{x\dot{y} - y\dot{x}}{x^2+y^2} = \frac{x\dot{y}-y\dot{x}}{r^2}.$$

6.5.1) a) We can write this system as:

$$\dot{x} = y, \quad \dot{y} = x^3 - x.$$

Thus, the equilibrium points are:

$$(x^*, y^*) = (-1,0), (0,0), (1,0).$$

We may compute:

$$A = \begin{bmatrix} 0 & 1 \\ 3x^2-1 & 0 \end{bmatrix}$$

At $(\pm 1, 0)$:

$$A = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \Rightarrow \lambda_{1,2} = \pm\sqrt{2}$$

$\Rightarrow$ Saddle Point

At $(0,0)$:

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \Rightarrow \lambda_{1,2} = \pm i$$

$\Rightarrow$ Center

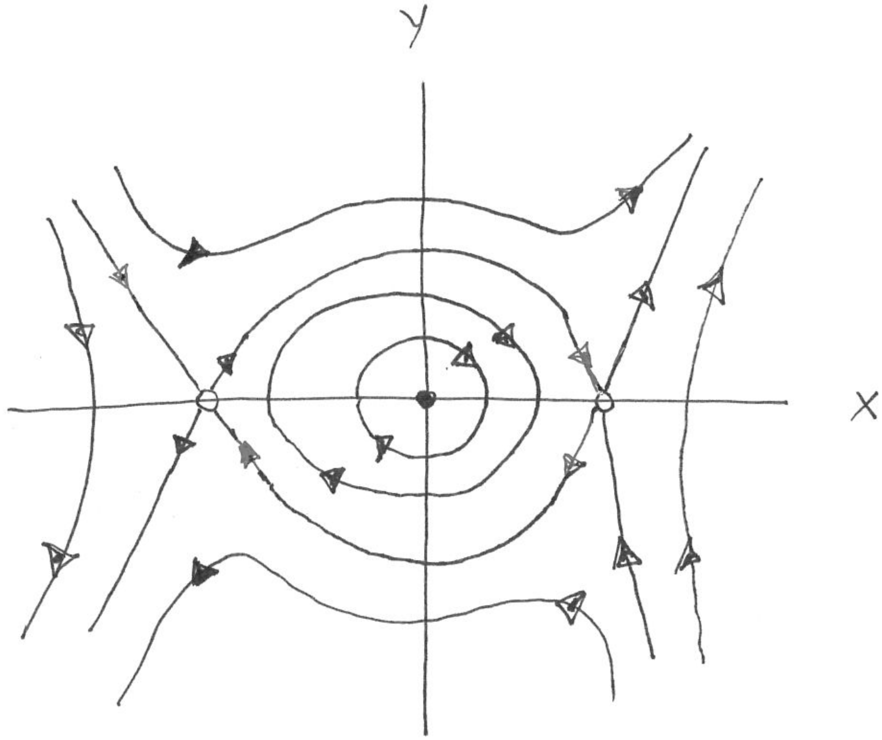
b) A conserved quantity is:

$$E = \frac{1}{2}y^2 - \frac{1}{4}x^4 + \frac{1}{2}x^2.$$

We may check:

$$\dot{E} = y\dot{y} - x^3\dot{x} + x\dot{x} = y(x^3 - x) - x^3 y + xy$$
$$= 0.$$

c) Thus, the phase portrait is:



6.5.20) b) $\frac{d}{dt}(P+R+S) = \dot{P}+\dot{R}+\dot{S}$

$$= P(R-S)+R(S-P)+S(P-R)$$

$$= PR - PS + RS - RP + SP - SR$$

$$= 0.$$

c) $\frac{d}{dt}(PRS) = \dot{P}RS + P\dot{R}S + PR\dot{S}$

$$= PRS(R-S) + PRS(S-P) + PRS(P-R)$$

$$= PRS(R-S+S-P+P-R)$$

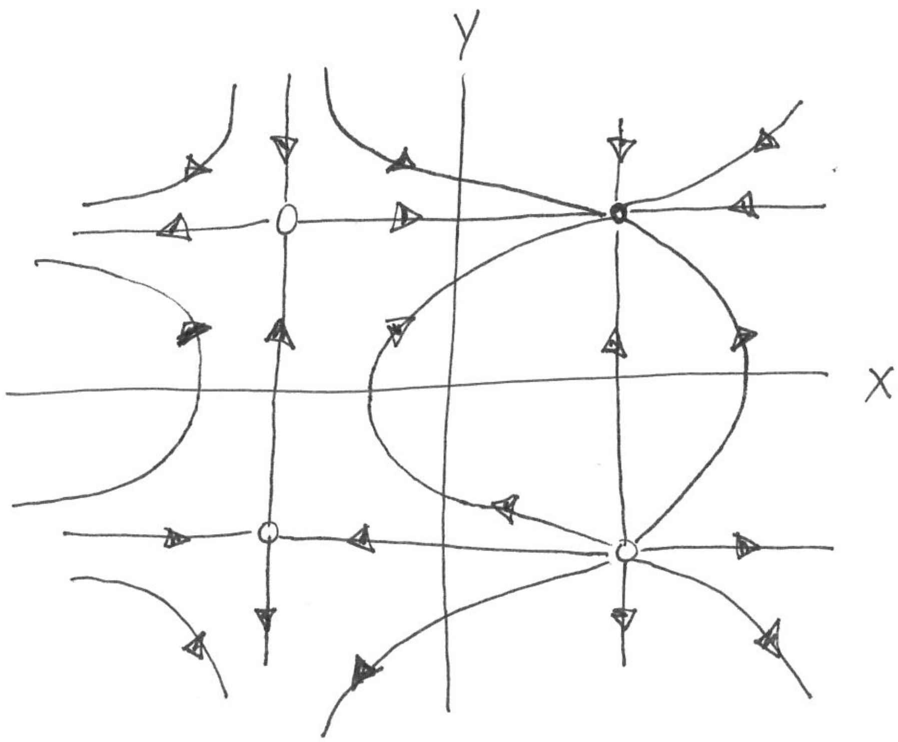
$$= 0.$$

6.6.1) Taking $t \to -t$, $y \to -y$:

$$-\dot{x} = -y(1-x^2) \Rightarrow \dot{x} = y(1-x^2),$$

$$-(-\dot{y}) = 1-(-y)^2 \Rightarrow \dot{y} = 1-y^2.$$

Thus, the system is reversible.



6.6.5) a) Under $t \to -t$, we have $\ddot{x} \to \ddot{x}$ and $\dot{x} \to -\dot{x}$, so:

$$\ddot{x} + f(\dot{x}) + g(x) = 0 \to \ddot{x} + f(-\dot{x}) + g(x) = 0$$

$$\Rightarrow \ddot{x} + f(\dot{x}) + g(x) = 0.$$