Homework 3 Solutions

4.1.1) The right-hand-side must be periodic with period 2π, so a must be an integer.

4.1.2) The system has fixed points where $1 + 2\cos\theta^* = 0 \Rightarrow \cos\theta^* = -\frac{1}{2}$

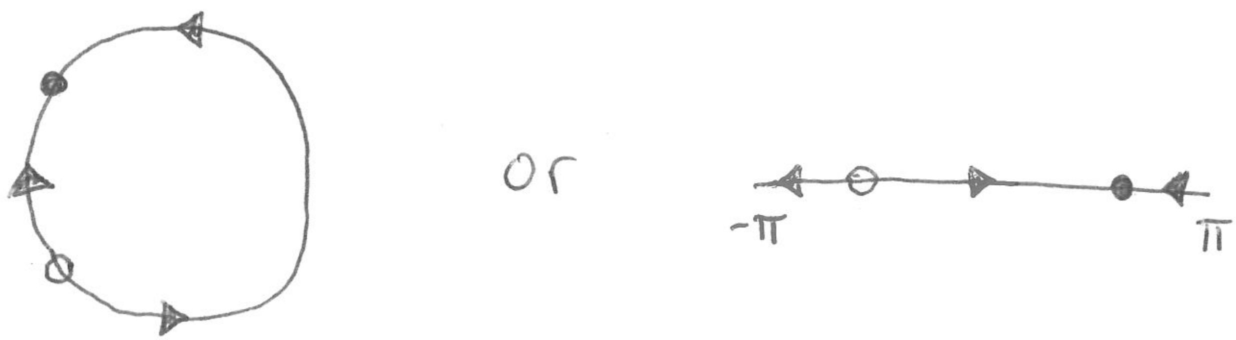
$$\Rightarrow \theta^* = \pm\frac{2\pi}{3}.$$

For stability, we have $f(\theta) = 1 + 2\cos\theta$ so $f'(\theta) = -2\sin\theta$.

$$f'\left(\frac{2\pi}{3}\right) = -\sqrt{3} \Rightarrow \text{stable.}$$

$$f'\left(-\frac{2\pi}{3}\right) = \sqrt{3} \Rightarrow \text{unstable.}$$

It is also acceptable to do this graphically. This gives the phase portrait:



4.2.1) Intuitively, the bells will ring atthe next time which is a common multiple of 3 and 4. The least common multiple is 12, so they will next ring together in 12 seconds.

Using the method in Example 4.2.1 yields:

$$T = \left(\frac{1}{T_1} - \frac{1}{T_2}\right)^{-1} = \left(\frac{1}{3} - \frac{1}{4}\right)^{-1} = \left(\frac{1}{12}\right)^{-1}$$

$$= 12 \text{ seconds.}$$

4.3.3) We have:

$$\dot{\theta} = \mu \sin\theta - \sin 2\theta$$

$$= \mu \sin\theta - 2\sin\theta\cos\theta$$

$$= \sin\theta\,(\mu - 2\cos\theta).$$

Thus, if $|\mu| \geq 2$, there are only two fixed points at $\theta^* = 0, \pi$.

If $|\mu| < 2$, there are additional fixed points at $\theta^* = \pm \cos^{-1}\left(\frac{\mu}{2}\right)$.

This gives three cases.

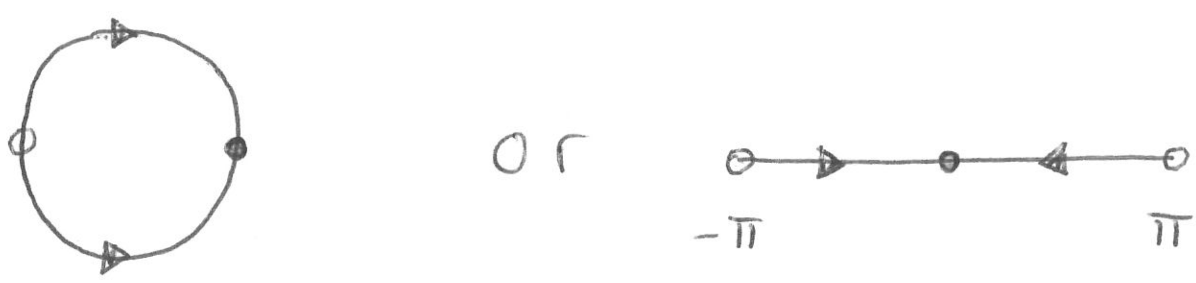
$\mu < -2$:

$$f(\theta) = \mu \sin\theta - \sin 2\theta$$

$$\Rightarrow f'(\theta) = \mu \cos\theta - 2\cos 2\theta.$$

$f'(0) = \mu - 2 < 0 \Rightarrow$ stable.

$f'(\pi) = -(\mu + 2) > 0 \Rightarrow$ unstable.



$-2 < \mu < 2$:

$f'(0) = \mu - 2 < 0 \Rightarrow$ stable.

$f'(\pi) = -(\mu+2) < 0 \Rightarrow$ stable.

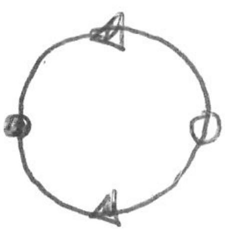
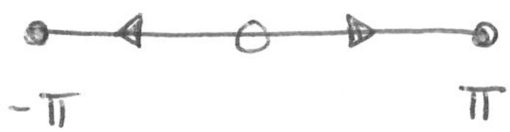
$$f'\left(\pm\cos^{-1}\left(\frac{\mu}{2}\right)\right) = 2 - \frac{\mu^2}{2} > 0 \Rightarrow \text{unstable.}$$

 or 

$\mu > 2$:

$$f'(0) = \mu - 2 > 0 \Rightarrow \text{unstable.}$$

$$f'(\pi) = -(\mu+2) < 0 \Rightarrow \text{stable.}$$

 or 

Thus, there are subcritical pitchfork bifurcations at $\mu = \pm 2$.

4.4.1) Let us express the differential equation in terms of a dimensionless time $\tau = t/T$:

$$mL^2\ddot{\theta} + b\dot{\theta} + mgL\sin\theta = \Gamma$$

$$\Rightarrow \frac{mL^2}{T^2}\theta'' + \frac{b}{T}\theta' + mgL\sin\theta = \Gamma$$

where $\theta' = \frac{d\theta}{d\tau}$. Since this is a balance of torques, let us divide by mgL to nondimensionalize:

$$\frac{L}{gT^2}\theta'' + \frac{b}{mgLT}\theta' + \sin\theta = \frac{\Gamma}{mgL}$$

We assume θ'' and θ' are $O(1)$ and we seek a time scale T so that:

$$\frac{b}{mgLT} = O(1) \quad \text{and} \quad \frac{L}{gT^2} << 1.$$

A natural choice is $T = \frac{b}{mgL}$. So:

$$\varepsilon\theta'' + \theta' + \sin\theta = \gamma$$

where $\varepsilon = \frac{gL^3m^2}{b^2}$ and $\gamma = \frac{\Gamma}{mgL}$.

Thus, the limit $\varepsilon \to 0$ is valid if

$$\varepsilon = \frac{gL^3m^2}{b^2} << 1 \Rightarrow b^2 >> gL^3m^2.$$

5.1.1) a) Method 1:

$$\frac{\dot{x}}{\dot{v}} = -\frac{v}{w^2 x} \Rightarrow w^2 x\dot{x} = -v\dot{v}$$

(Integrate both sides)

$$\Rightarrow \frac{w^2}{2}x^2 = -\frac{1}{2}v^2 + c$$

$$\Rightarrow w^2x^2 + v^2 = C.$$

Method 2: $\frac{d}{dt}(w^2x^2 + v^2) = 2w^2x\dot{x} + 2v\dot{v}$

$$= 2w^2xv - 2wxv = 0$$

So, $w^2x^2 + v^2 = C$.

b) We have $\frac{w^2x^2}{2}$ being the classical potential energy and $\frac{v^2}{2}$ being the kinetic energy (with mass 1). Thus the total energy $\frac{w^2x^2}{2} + \frac{v^2}{2}$ is conserved.

5.1.2) We have

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = -\frac{y}{ax} \text{ and } \begin{cases} x = c_1 e^{at} \\ y = c_2 e^{-t} \end{cases}$$

As $t \to \infty$:

$$\frac{dy}{dx} = -\frac{c_2 e^{-t}}{a c_1 e^{at}} = -\frac{c_2}{a c_1} e^{-(a+1)t} \to \pm\infty$$

since $a+1<0$.

So, all trajectories become parallel to the y-axis.

As $t \to -\infty$:

$$\frac{dy}{dx} = -\frac{c_2}{a c_1} e^{-(a+1)t} \longrightarrow 0 \text{ since } a+1<0.$$

So, all trajectories become parallel to the x-axis.

5.2.1) a) $A = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$.

$$\Rightarrow \det(A-\lambda I) = \det\begin{bmatrix} 4-\lambda & -1 \\ 2 & 1-\lambda \end{bmatrix}$$

$$= (4-\lambda)(1-\lambda) + 2$$

$$= (\lambda^2 - 5\lambda + 4) + 2$$

$$= \lambda^2 - 5\lambda + 6.$$

$\Rightarrow$ E-values occur when

$$\lambda^2 - 5\lambda + 6 = 0$$

$$\Rightarrow (\lambda-3)(\lambda-2) = 0$$

$$\Rightarrow \lambda = 3, 2.$$

$$\lambda = 3: \quad A - \lambda I = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}$$

$$\Rightarrow \bar{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

$$\lambda = 2: \quad A - \lambda I = \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix}$$

$$\Rightarrow \bar{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

b) The general solution is

$$\bar{x}(t) = c_1 e^{3t}\begin{bmatrix}1\\1\end{bmatrix} + c_2 e^{2t}\begin{bmatrix}1\\2\end{bmatrix}$$

$$= \begin{bmatrix} c_1 e^{3t} + c_2 e^{2t} \\ c_1 e^{3t} + 2c_2 e^{2t}\end{bmatrix}.$$

c) The origin is an unstable node.

d) The constants c_1 + c_2 solve

$$\begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}3\\4\end{bmatrix} \Rightarrow \begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix}$$

So, $\bar{x}(t) = \begin{bmatrix} 2e^{3t} + e^{2t} \\ 2e^{3t} + 2e^{2t}\end{bmatrix}.$

5.2.2) a) $A = \begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}.$

$$\Rightarrow \det(A - \lambda I) = \det\begin{bmatrix}1-\lambda & -1\\1 & 1-\lambda\end{bmatrix}$$

$$= (1-\lambda)(1-\lambda)+1$$

$$= (\lambda^2 - 2\lambda + 1) + 1$$

$$= \lambda^2 - 2\lambda + 2$$

$\Rightarrow$ E-values occur when

$$\lambda^2 - 2\lambda + 2 = 0$$

$$\Rightarrow \lambda = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i.$$

$$\lambda = 1+i: \quad A - \lambda I = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}$$

$$\Rightarrow \bar{v} = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

$$\lambda = 1-i: \quad A - \lambda I = \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix}$$

$$\Rightarrow \bar{v} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

b) $$\bar{x}(t) = c_1 e^{(1+i)t}\begin{bmatrix} i \\ 1 \end{bmatrix} + c_2 e^{(1-i)t}\begin{bmatrix} -i \\ 1 \end{bmatrix}$$

$$= e^t \begin{bmatrix} ic_1 e^{it} - ic_2 e^{-it} \\ c_1 e^{it} + c_2 e^{-it} \end{bmatrix}$$

$$= e^t \begin{bmatrix} ic_1(\cos t + i\sin t) - ic_2(\cos t - i\sin t) \\ c_1(\cos t + i\sin t) + c_2(\cos t - i\sin t) \end{bmatrix}$$

$$= e^t \begin{bmatrix} -(c_1+c_2)\sin t + i(c_1-c_2)\cos t \\ (c_1+c_2)\cos t + i(c_1-c_2)\sin t \end{bmatrix}$$

$$= e^t \begin{bmatrix} -d_1 \sin t + d_2 \cos t \\ d_1 \cos t + d_2 \sin t \end{bmatrix}$$

Where $d_1 = c_1 + c_2$ and $d_2 = i(c_1 - c_2)$.

5.2.8) $A = \begin{bmatrix} -3 & 4 \\ -2 & 3 \end{bmatrix}$.

$$\Rightarrow \det(A - \lambda I) = \det \begin{bmatrix} -3-\lambda & 4 \\ -2 & 3-\lambda \end{bmatrix}$$

$$= (-3-\lambda)(3-\lambda) + 8$$

$$= (\lambda^2 - 9) + 8 = \lambda^2 - 1.$$

$$\Rightarrow \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1.$$

$$\lambda = 1: \quad A - \lambda I = \begin{bmatrix} -4 & 4 \\ -2 & 2 \end{bmatrix}$$

$$\Rightarrow \bar{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda = -1: \quad A - \lambda I = \begin{bmatrix} -2 & 4 \\ -2 & 4 \end{bmatrix}$$

$$\Rightarrow \bar{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The origin is a saddle point.