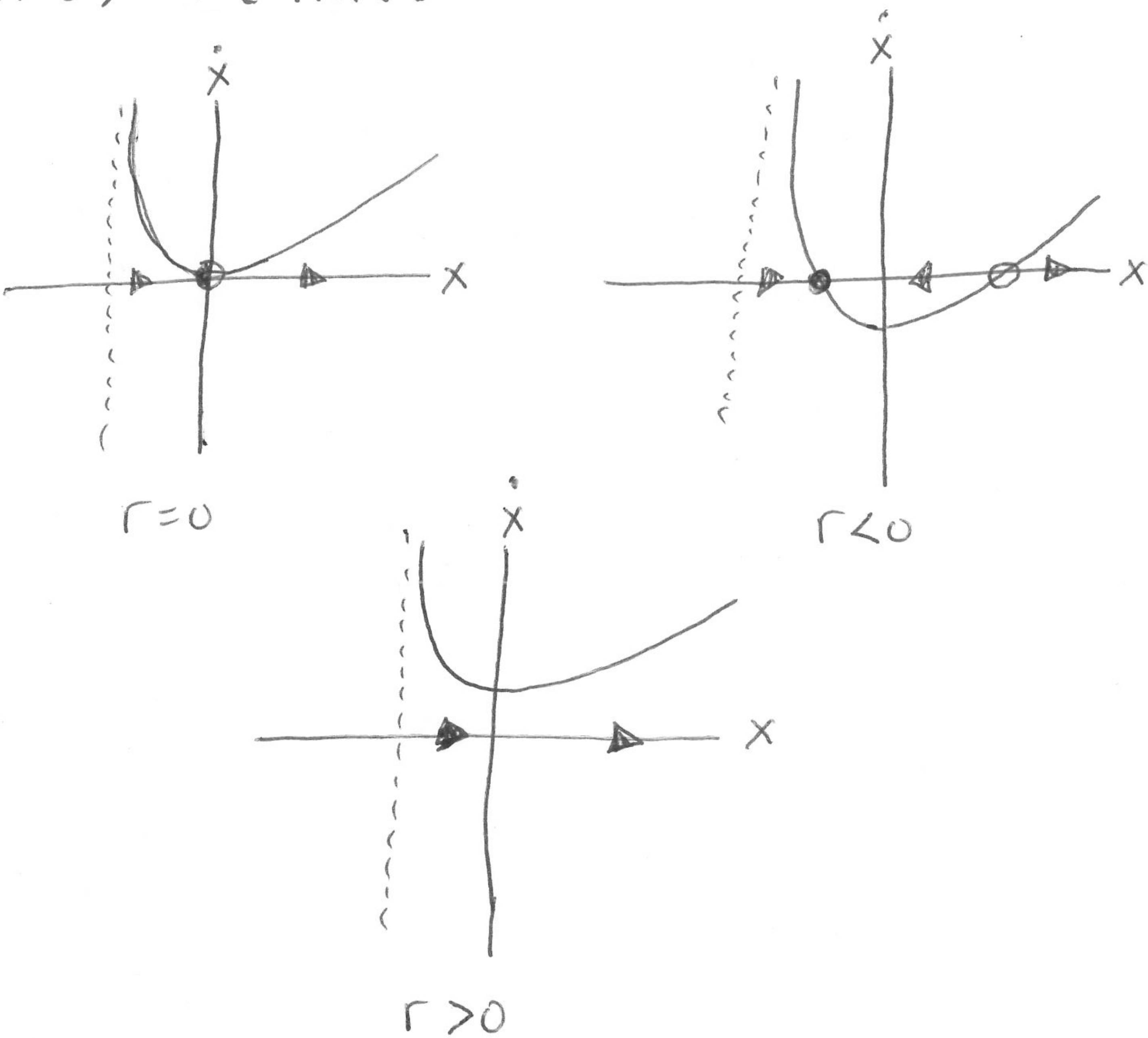
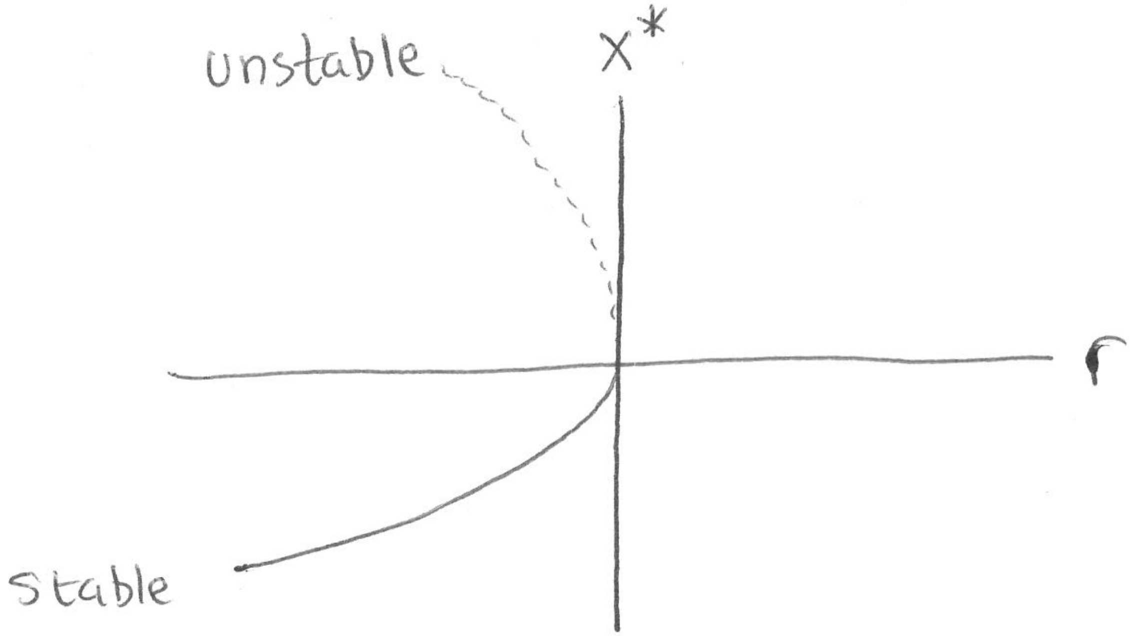
Homework 2 Solutions

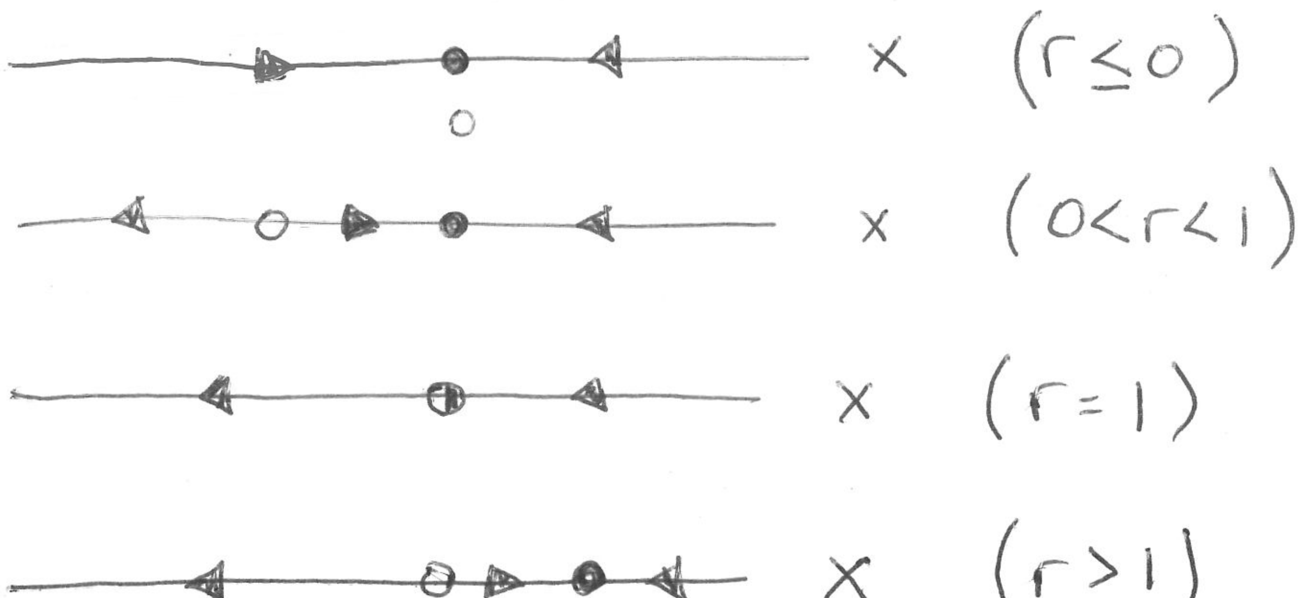
3.1.3) We have:



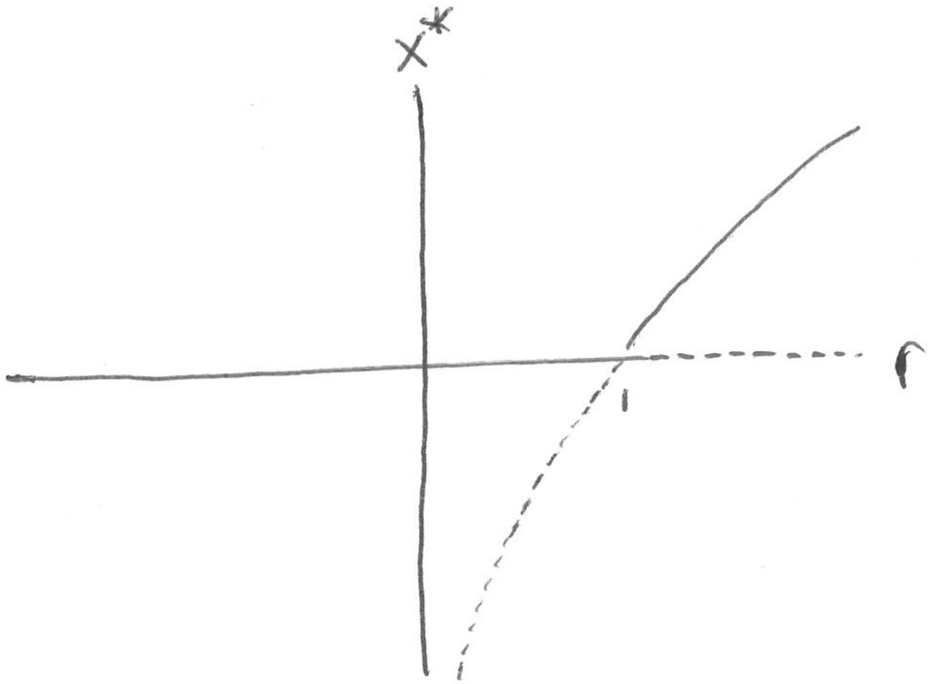
Thus, there is a saddle-node bifurcation at $r=0$. The bifurcation diagram looks like:



3.2.4) We have:



So, there is a transcritical bifurcation at $r=1$. The bifurcation diagram looks like:



3.3.2) a) If $\dot{P} = 0$ and $\dot{D} = 0$, we have:

$$\begin{cases} 0 = \gamma_1 (ED - P), \\ 0 = \gamma_2 (\lambda + 1 - D - \lambda E P). \end{cases}$$

So:

$$\begin{cases} P = \dfrac{E(1+\lambda)}{1+\lambda E^2}, \\ D = \dfrac{1+\lambda}{1+\lambda E^2}. \end{cases}$$

Substituting into $\dot{E} = k(P - E)$, we find:

$$\dot{E} = k\left(\frac{E(1+\lambda)}{1+\lambda E^2} - E\right) = \lambda k \frac{E(1-E^2)}{1+\lambda E^2}$$

b) Solving $\dot{E}=0$, we find the fixed points $E^*=0,\pm 1$ when $\lambda \neq 0$ and all E when $\lambda = 0$.

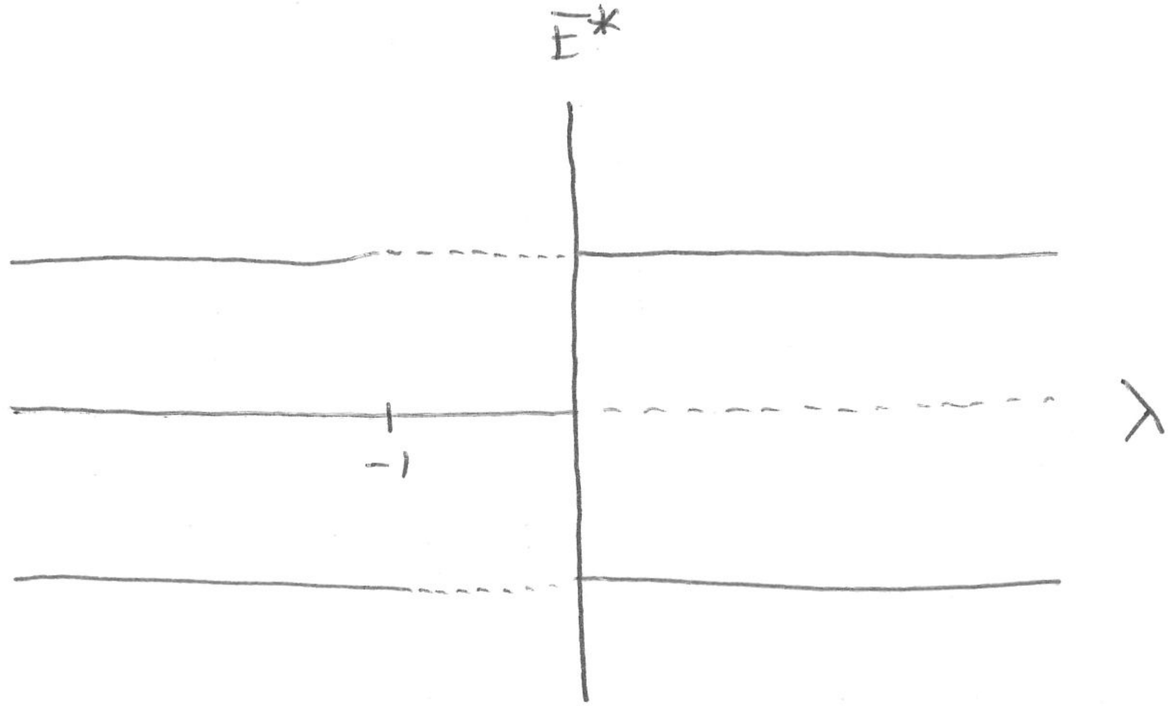
c) If $f(E) = \lambda K \frac{E(1-E^2)}{1+\lambda E^2}$, we have

$$f'(E) = \lambda K \frac{1-E^2(3+\lambda)-E^4\lambda}{(1+\lambda E^2)^2}.$$

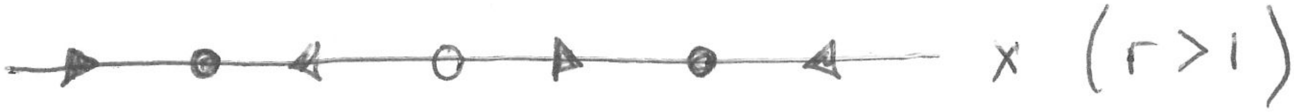
At $E^*=0$: $f'(0)=\lambda K$, so the fixed point is stable if $\lambda<0$ and unstable if $\lambda>0$.

At $E^*=\pm 1$: $f'(\pm 1) = \frac{-2K\lambda}{1+\lambda}$, so the fixed point is stable if $\lambda>0$, unstable if $-1<\lambda<0$, and stable if $\lambda<-1$.

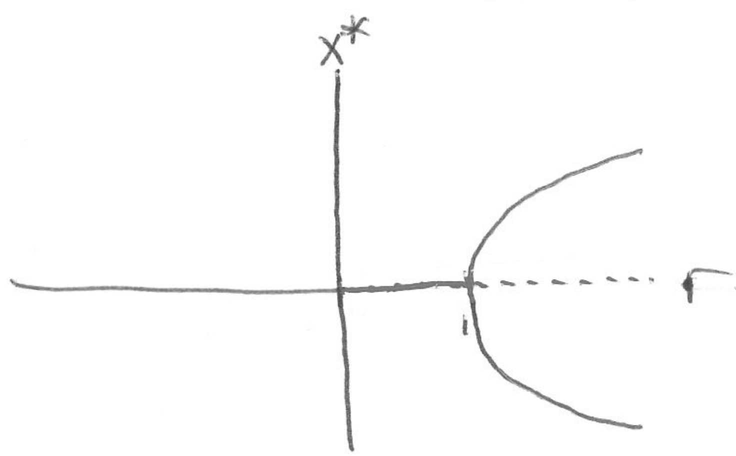
Thus, the bifurcation diagram looks like:



3.4.2) We have:



So, there is a supercritical pitchfork bifurcation at $r = 1$. The bifurcation diagram looks like:



3.4.12) Method 1:

Arrange that multiple saddle-node bifurcations occur simultaneously. For example, $\dot{x} = r - (x-1)^2(x+1)^2$ has no fixed points for $r<0$. When $r>0$, there are four branches of fixed points. The saddle-node bifurcations occur at $r=0$, $x=\pm 1$.

Method 2:

Consider $\dot{x} = (r-x^2)(2r-x^2)$. There are no fixed points for $r<0$ and four branches of fixed points when $r>0$. These occur at $x=\pm\sqrt{r}, \pm\sqrt{2r}$. All four fixed points emerge at $r=0$, $x=0$.

3.5.2) Equation 3.5.7 reads:

$$\frac{d\phi}{d\tau} = \sin\phi(\gamma\cos\phi - 1).$$

For $\gamma < 1$: The only fixed point is $\phi^* = 0$.

$$f(\phi) = \sin\phi(\gamma\cos\phi - 1)$$

$$\Rightarrow f'(\phi) = \gamma\cos 2\phi - \cos\phi.$$

So, $f'(0) = \gamma - 1 < 0 \Rightarrow \phi^* = 0$ is stable.

For $\gamma > 1$: The fixed points are $\phi^* = 0$ and $\phi^* = \pm\cos^{-1}\frac{1}{\gamma}$.

$f'(0) = \gamma - 1 > 0 \Rightarrow \phi^* = 0$ is unstable.

$f'\left(\pm\cos^{-1}\frac{1}{\gamma}\right) = \frac{1}{\gamma} - \gamma < 0 \Rightarrow \phi^* = \pm\cos^{-1}\frac{1}{\gamma}$ is stable.

3.5.3) Near $\phi = 0$, we have:

$\sin\phi = \phi - \frac{\phi^3}{3!} + O(\phi^5)$ and

$\cos\phi = 1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} + O(\phi^5)$.

So, $f(\phi) = \left(\phi - \frac{\phi^3}{6} + O(\phi^5)\right)$

$$\left(\gamma\left(1 - \frac{\phi^2}{2} + \frac{\phi^4}{24} + O(\phi^5)\right) - 1\right)$$

$$= (\gamma-1)\phi - \left(\frac{\gamma-1}{6} + \frac{\gamma}{2}\right)\phi^3 + O(\phi^5)$$

$$= (\gamma-1)\phi + \frac{1-4\gamma}{6}\phi^3 + O(\phi^5)$$

$$\Rightarrow A = \gamma - 1, \quad B = \frac{1-4\gamma}{6}.$$

3.5.8) We have:

$$\frac{du}{dt} = au + bu^3 - cu^5.$$

Taking $x = \frac{u}{U}$, $\tau = \frac{t}{T}$ yields:

$$\frac{U}{T}\frac{dx}{d\tau} = aUx + bU^3x^3 - cU^5x^5$$

$$\Rightarrow \frac{dx}{d\tau} = aTx + bU^2Tx^3 - cU^4Tx^5.$$

We seek U, T so that $bU^2T = 1$, $cU^4T = 1$.

Choose $U = \sqrt{\frac{b}{c}}$, $T = \frac{c}{b^2}$.

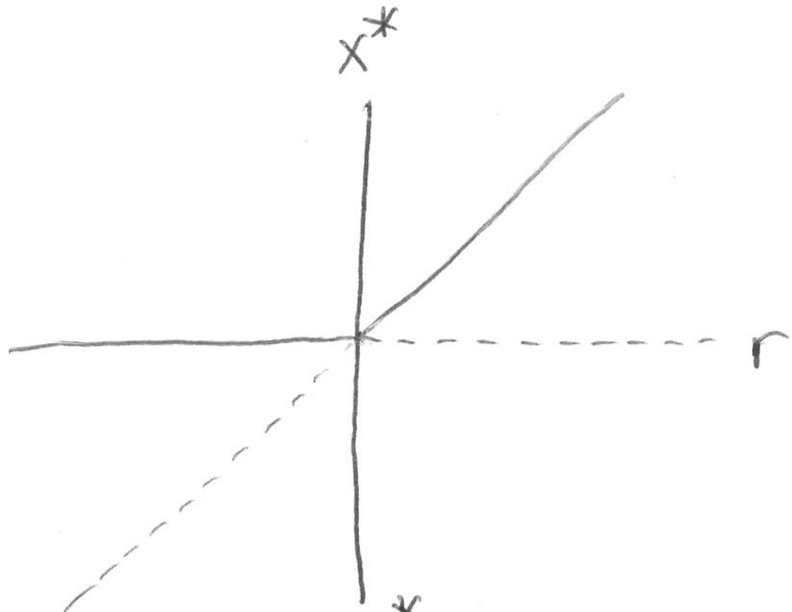
Then:

$$\frac{dx}{d\tau} = \frac{ac}{b^2} x + x^3 - x^5$$

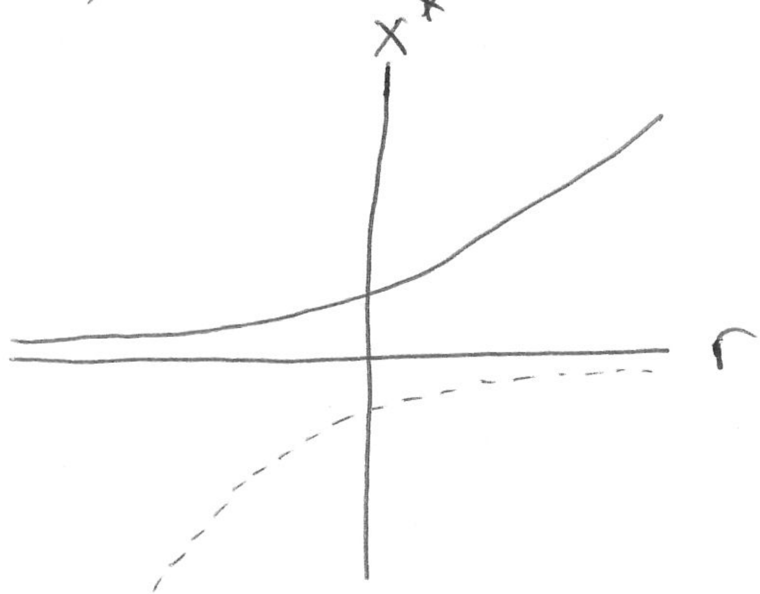
$$= r x + x^3 - x^5 \text{ where } r = \frac{ac}{b^2}.$$

3.6.2) a)

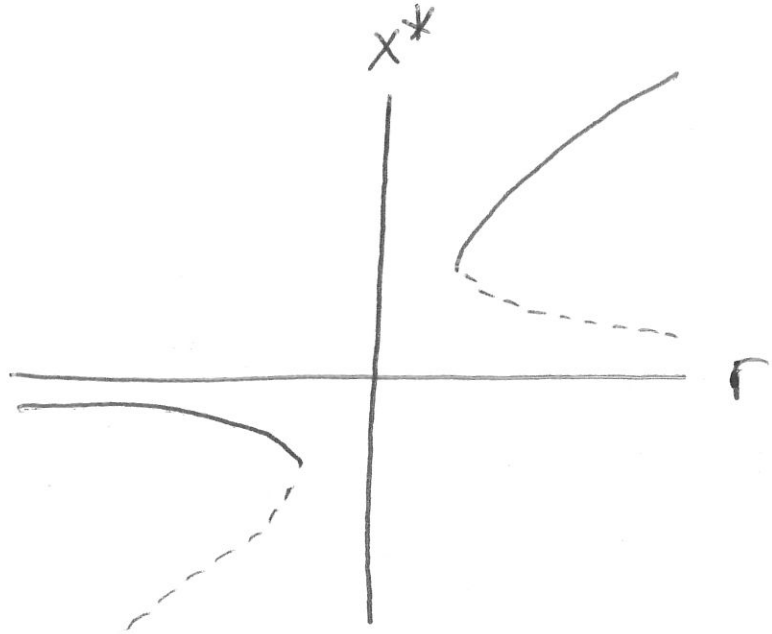
For $h = 0$:



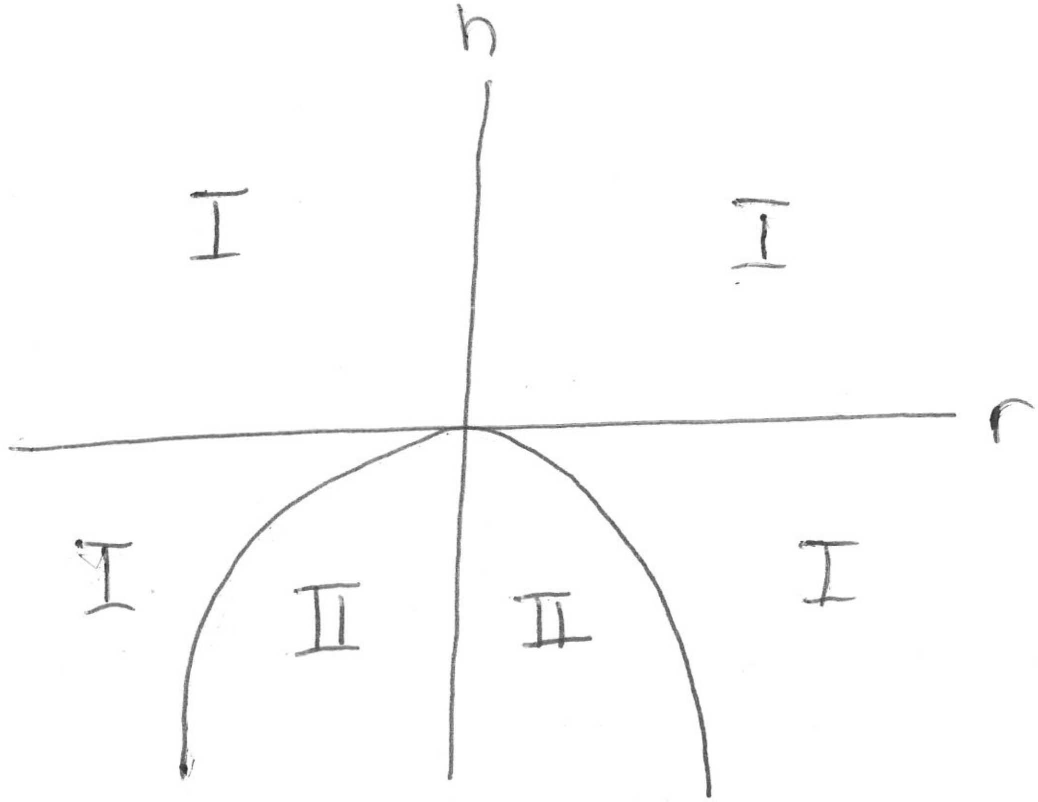
For $h > 0$:



For $h < 0$:



b)



In region I: Two fixed points, one unstable, one stable.

Region II: No fixed points.

On the boundary between I + II:
One fixed point, half-stable.

There is a saddle node bifurcation crossing from region I to II, except at (0,0) where there is the transcritical bifurcation observed before.