Homework 1 Solutions

$\boxed{2.1.1}$ Fixed points of $\dot{x} = \sin x$ solve $\sin x = 0$, so $x^* = k\pi$ for $k \in \mathbb{Z}$.

$\boxed{2.1.2}$ At maxima of $\dot{x} = \sin x$, so $x = \frac{\pi}{2} + k2\pi$ for $k \in \mathbb{Z}$.

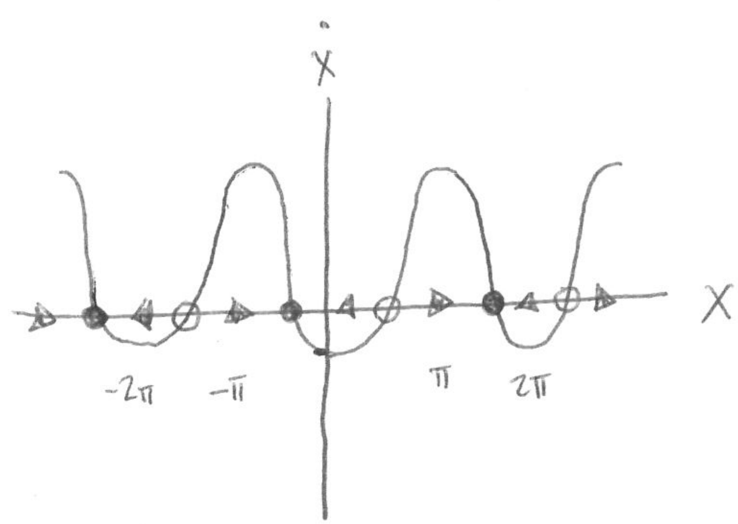
$\boxed{2.1.3}$ a) $\ddot{x} = \frac{d}{dt}(\dot{x}) = \frac{d}{dt}(\sin x) = \dot{x}\cos x$

$$= \sin x \cos x$$

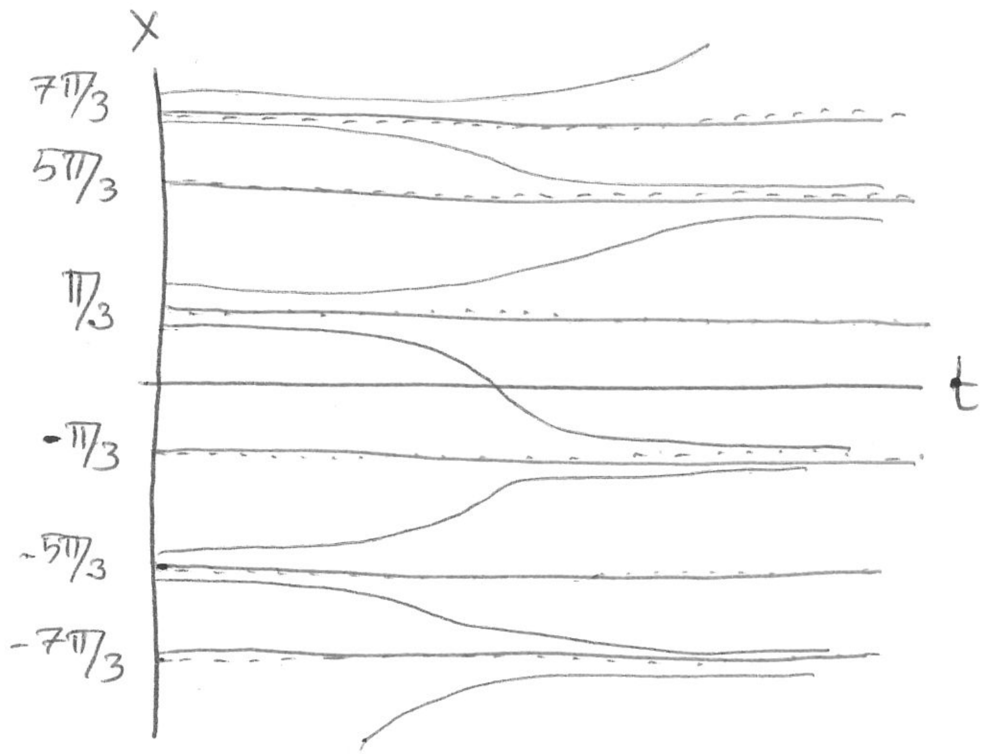
$$= \frac{1}{2}\sin 2x$$

b) At maxima of $\ddot{x} = \frac{1}{2}\sin 2x$, so $x = \frac{\pi}{4} + k\pi$ for $k \in \mathbb{Z}$.

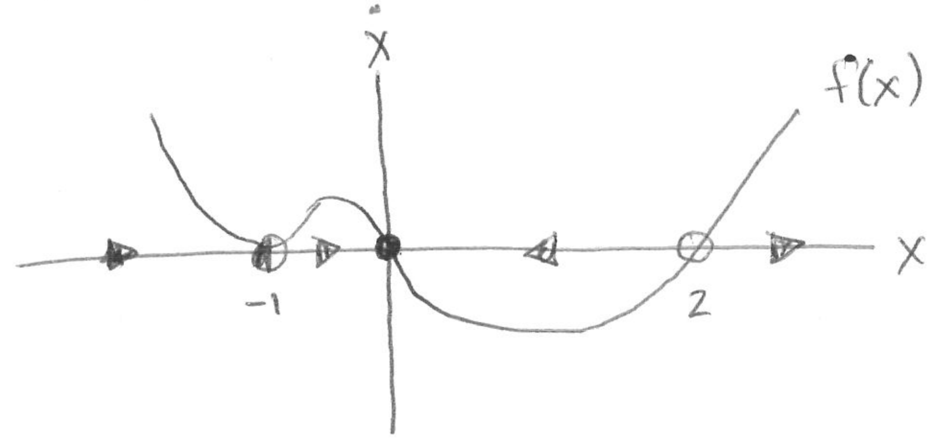
$\boxed{2.2.6}$ $\dot{x} = 1 - 2\cos x$



Fixed points are $x^* = \frac{\pi}{3} + 2\pi k$, which are unstable, and $x^* = -\frac{\pi}{3} + 2\pi k$, which are stable.



$\boxed{2.2.8}$ You need the phase portrait to look like:



$\Rightarrow$ Double root at $x=-1$, single roots at $x=0$ and $x=2$.

One possible solution is:

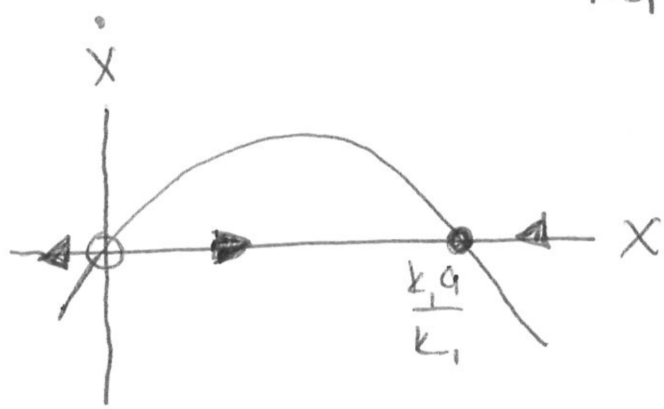
$$\dot{x} = (x+1)^2(x)(x-2).$$

$\boxed{2.3.2}$ a) We may factor:

$$\dot{x} = k_{-1}\, x \left(\frac{k_1 a}{k_{-1}} - x \right).$$

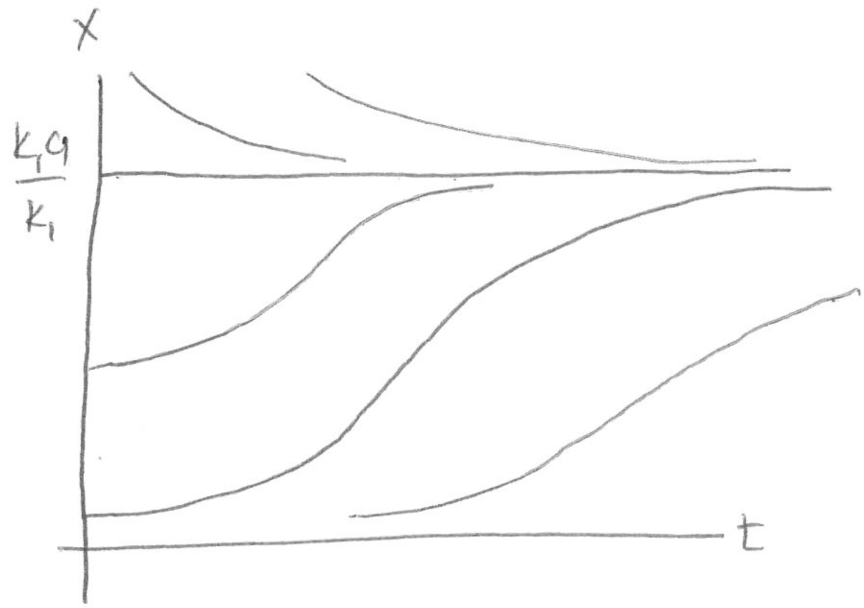
So, there are fixed points at

$$x^* = 0 \text{ and } x^* = \frac{k_1 a}{k_{-1}}.$$



$\Rightarrow$ $x^* = 0$ is unstable, $x^* = \frac{k_1 a}{k_{-1}}$ is stable.

b)



$\boxed{2.4.7}$ If $a \geq 0$, we may factor:

$$\dot{x} = x(\sqrt{a}+x)(\sqrt{a}-x)$$

Otherwise, $\dot{x} = x(a-x^2)$.

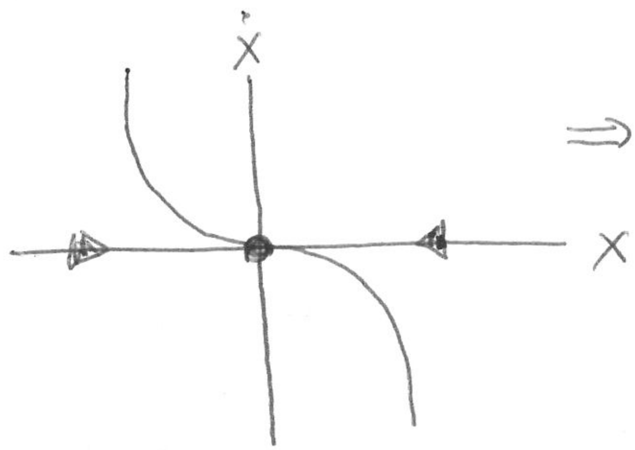
$a<0$: $f'(x) = a-3x^2$

$x^* = 0 \Rightarrow f'(x^*) = a < 0$

$\Rightarrow$ stable!

$a=0$:

$x^* = 0 \Rightarrow f'(x^*) = a = 0$



$\Rightarrow$ stable!

$a>0$:

$x^* = 0 \Rightarrow f'(x^*) = a > 0 \Rightarrow$ unstable!

$x^* = \sqrt{a} \Rightarrow f'(x^*) = -2a < 0 \Rightarrow$ stable!

$x^* = -\sqrt{a} \Rightarrow f'(x^*) = -2a < 0 \Rightarrow$ stable!

$\boxed{2.5.4}$ The solution is:

$$x(t) = \begin{cases} 0 & t \le t_0 \\ \left(\frac{2}{3}(t-t_0)\right)^{3/2} & t > t_0 \end{cases}$$

for any $t_0 \ge 0$.

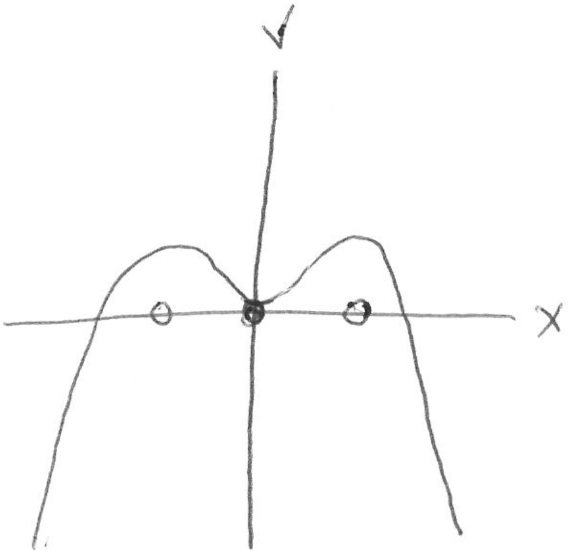
$\boxed{2.6.1}$ The system $m\ddot{x} = -kx$ is actually a two-dimensional system in the language of dynamical systems.

$\boxed{2.7.6}$ The interesting ranges of r are

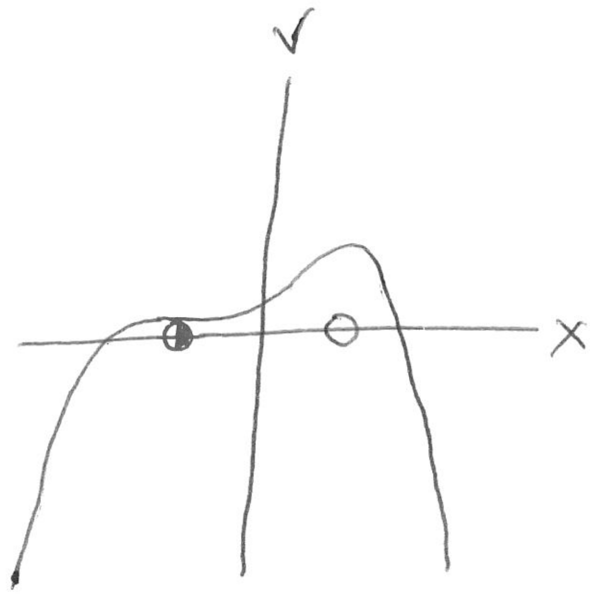
$$|r| < \frac{2}{3\sqrt{3}}, \quad |r| = \frac{2}{3\sqrt{3}}, \quad |r| > \frac{2}{3\sqrt{3}}.$$

Since the potential function is $rx + \frac{1}{2}x^2 - \frac{1}{4}x^4 = V(x)$ we have the following cases:

$-\frac{2}{3\sqrt{3}} < r < \frac{2}{3\sqrt{3}}$:

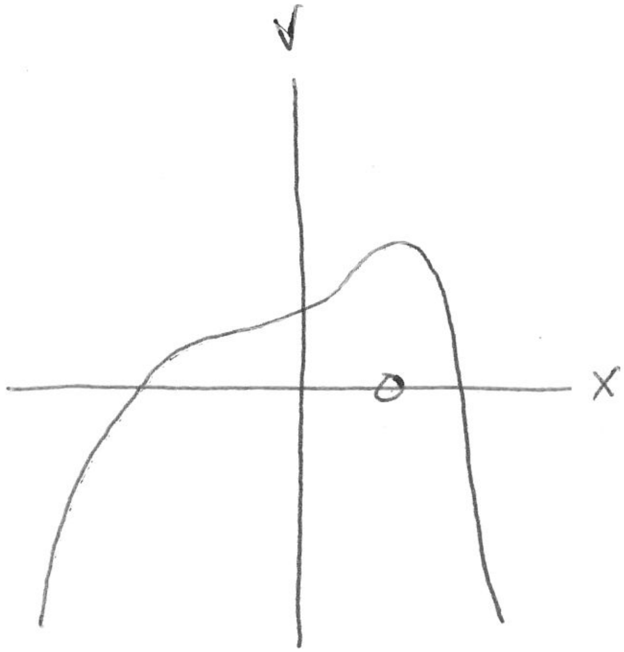


$r = \frac{2}{3\sqrt{3}}$:



(mirrored if $r = \frac{-2}{3\sqrt{3}}$)

$r > \frac{2}{3\sqrt{3}}$:



(mirrored if $r < \frac{-2}{3\sqrt{3}}$)